On the concentration of semi-classical states for a nonlinear Dirac-Klein-Gordon system

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Abstract

In the present paper, we study the semi-classical approximation of a Yukawa-coupled massive Dirac-Klein-Gordon system with some general nonlinear self-coupling. We prove that for a constrained coupling constant there exists a family of ground states of the semi-classical problem, for all $\hbar$ small, and show that the family concentrates around the maxima of the nonlinear potential as $\hbar \to 0$. Our method is variational and relies upon a delicate cutting off technique. It allows us to overcome the lack of convexity of the nonlinearities.

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1   Introduction and main result

In this paper we study the solitary wave solutions of the massive Dirac-Klein-Gordon system involving an external self-coupling:

$$
(1.1) \quad
\begin{cases}
i\dfrac{\hbar}{c}\,\partial_t\psi + i\hbar\displaystyle\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc\beta\psi - \lambda\phi\beta\psi = f(x,\psi) \\[2mm]
\dfrac{\hbar^2}{c^2}\,\partial_t^2\phi - \hbar^2\Delta\phi + M\phi = 4\pi\lambda(\beta\psi)\cdot\psi
\end{cases}
$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^3$, where c is the speed of light, $\hbar$ is Planck's constant, $\lambda > 0$ is coupling constant, m is the mass of the electron and M is the mass of the meson (we use the notation $u \cdot v$ to express the inner product of $u,v \in \mathbb{C}^4$). Here α_1, α_2, α_3 and β are 4×4 complex Pauli matrices:

$$
\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad
\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3,
$$

with
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

System (1.1) arises in mathematical models of particle physics, especially in nonlinear topics. Physically, system (1.1) describes the Dirac and Klein-Gordon equations coupled through the Yukawa interaction between a Dirac field $\psi \in \mathbb{C}^4$ and a scalar field $\phi \in \mathbb{R}$ (see [6]). This system is inspired by approximate descriptions of the external force involve only functions of fields. The nonlinear self-coupling $f(x, \psi)$, which describes a self-interaction in Quantum electrodynamics, gives a closer description of many particles found in the real world. Various nonlinearities are considered to be possible basis models for unified field theories (see [20], [21], [23] etc. and references therein).

System (1.1) with null external self-coupling, i.e., $f \equiv 0$, has been studied for a long time and results are available concerning the Cauchy problem (see [7], [8], [9], [25], [28] etc.). The first result on the global existence and uniqueness of solutions of (1.1) (in one space dimension) was obtained by J. M. Chadam in [8] under suitable assumptions on the initial data. For later developments, we mention, e.g., that J. M. Chadam and Robert T. Glassey [9] yield the existence of a global solution in three space dimensions. In [7], N. Bournaveas obtained low regularity solutions of the Dirac-Klein-Gordon system by using classical Strichartz-type time-space estimates.

As far as the existence of stationary solutions (solitary wave solutions) of (1.1) is concerned, there is a pioneering work by M. J. Esteban, V. Georgiev and E. Séré (see [19]) in which a multiplicity result is studied. Here, by stationary solution, we mean a solution of the type

(1.2)
$$\begin{cases} \psi(t, x) = \varphi(x)e^{-i\xi t/\hbar}, \quad \xi \in \mathbb{R}, \quad \varphi : \mathbb{R}^3 \to \mathbb{C}^4, \\ \phi = \phi(x). \end{cases}$$

In [19], using the variational arguments, the authors obtained infinitely many solutions for $\xi \in (-\frac{mc}{\hbar}, 0)$ under the assumption

$$\varphi(x) = \begin{pmatrix} v(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r) & \begin{pmatrix} \cos \vartheta \\ e^{i\tau} \sin \vartheta \end{pmatrix} \end{pmatrix}$$

where (r, ϑ, τ) are the spherical coordinates of $x \in \mathbb{R}^3$.

We emphasize that the works mentioned above were mainly concerned with the autonomous system with null self-coupling. Besides, limited work has been done in the semi-classical approximation. In the present paper we are devoted to the existence and concentration phenomenon of stationary semi-classical solutions to system (1.1). For small $\hbar$, the solitary waves are

referred to as semi-classical states. To describe the transition from quantum to classical mechanics, the existence of solutions $(\varphi_\hbar, \phi_\hbar)$, $\hbar$ small, possesses an important physical interest. More precisely, for ease of notations, denoted by $\varepsilon = \hbar$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, we are concerned with (substitute (1.2) in (1.1)) the following stationary nonlinear Dirac-Klein-Gordon system:

(1.3)
$$\begin{cases} i\varepsilon\,\alpha \cdot \nabla\varphi - a\beta\varphi + \omega\varphi - \lambda\phi\beta\varphi = W(x)g(|\varphi|)\varphi\,, \\ -\,\varepsilon^2\Delta\phi + M\phi = 4\pi\lambda(\beta\varphi) \cdot \varphi\,. \end{cases}$$

where $a = mc > 0$ and $\omega \in \mathbb{R}$.

On the nonlinear self-coupling, writing $G(|w|) := \int_0^{|w|} g(s)s\,ds$, we make the following hypotheses:

(P_0) $W \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with $\inf W > 0$ and $\limsup_{|x|\to\infty} W(x) < \max W(x)$;

(G_1) $g(0) = 0$, $g \in C^1(0, \infty)$, $g'(s) > 0$ for $s > 0$, and there exist $p \in (2, 3)$, $c_1 > 0$ such that $g(s) \le c_1(1 + s^{p-2})$ for $s \ge 0$;

(G_2) there exist $\sigma > 2$, $\theta > 2$ and $c_0 > 0$ such that $c_0 s^\sigma \le G(s) \le \frac{1}{\theta}g(s)s^2$ for all $s > 0$.

A typical example is the power function $g(s) = s^{\sigma-2}$.

For showing the concentration phenomena, we set $m := \max_{x\in\mathbb{R}^3} W(x)$ and
$$\mathscr{C} := \{x \in \mathbb{R}^3 : W(x) = m\}.$$

Our result reads as

Theorem 1.1. *Assume that* $\omega \in (-a, a)$, (P_0) *and* (G_1)-(G_2) *are satisfied. Then there exists* $\lambda_0 > 0$ *such that given* $\lambda \in (0, \lambda_0]$, *for all* $\varepsilon > 0$ *small,*

1. *The system* (1.3) *possesses at least one ground state solution* $(\varphi_\varepsilon, \phi_\varepsilon) \in \cap_{q\ge2}W^{1,q}(\mathbb{R}^3, \mathbb{C}^4) \times C^2(\mathbb{R}^3, \mathbb{R})$.

2. *The set of ground state solutions is compact in* $H^1(\mathbb{R}^3, \mathbb{C}^4) \times H^1(\mathbb{R}^3, \mathbb{R})$.

3. *If additionally* ∇W *is bounded, then*

   (i) *There is a maximum point* x_ε *of* $|\varphi_\varepsilon|$ *with* $\lim_{\varepsilon\to0} \text{dist}(x_\varepsilon, \mathscr{C}) = 0$ *such that the pair* $(u_\varepsilon, V_\varepsilon)$, *where* $u_\varepsilon(x) := \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$ *and* $V_\varepsilon := \phi_\varepsilon(\varepsilon x + x_\varepsilon)$, *converges in* $H^1 \times H^1$ *to a ground state solution of (the limit equation)*

   (1.4)
   $$\begin{cases} i\alpha \cdot \nabla u - a\beta u + \omega u - \lambda V\beta u = mg(|u|)u \\ -\,\Delta V + MV = 4\pi\lambda(\beta u) \cdot u \end{cases}.$$

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(ii) $|\varphi_\varepsilon(x)| \leq C \exp\left(-\frac{c}{\varepsilon}|x - x_\varepsilon|\right)$ *for some* $C, c > 0$.

It is standard that (1.3) is equivalent to, by letting $u(x) = \varphi(\varepsilon x)$ and $V(x) = \phi(\varepsilon x)$,

$$(1.5) \qquad \begin{cases} i\alpha \cdot \nabla u - a\beta u + \omega u - \lambda V\beta u = W_\varepsilon(x)g(|u|)u \\ -\Delta V + M\,V = 4\pi\lambda(\beta u)\cdot u \end{cases}$$

where $W_\varepsilon(x) = W(\varepsilon x)$. We will in the sequel focus on this equivalent problem. Our proofs are variational: the semiclassical solutions that are obtained as critical points of an energy functional Φ_ε associated to the equivalent problem (1.5).

There have been a large number of works on existence and concentration phenomenon of semi-classical states of nonlinear Schrödinger-Poisson systems arising in the *non-relativistic* quantum mechanics, see, for example, [2, 3, 4] and their references. And, only very recently, the papers [16, 17] studied the existence of a family of semi-classical ground states of Maxwell-Dirac system and showed that the family concentrates around some certain sets as $\varepsilon \to 0$. It is quite natural to ask if certain similar results can be obtained for nonlinear Dirac-Klein-Gordon systems arising in the relativistic quantum mechanics. Mathematically, the problems in Dirac-Klein-Gordon systems are difficult because they are strongly indefinite in the sense that both the negative and positive parts of the spectrum of Dirac operator are unbounded and consist of essential spectrums.

It should be pointed out that Ding, jointly with co-authors, developed some technique arguments to obtain the existence and concentration of semi-classical solutions for nonlinear Dirac equations (not for Dirac-Klein-Gordon system), see [12, 13, 14]. Compared with the papers, difficulty arises in the Dirac-Klein-Gordon system because of the presence of the action for a meson field ϕ. In order to overcome this obstacle, we develop a cut-off arguments. Roughly speaking, an accurate uniformly boundedness estimates on $(C)_c$ (Cerami) sequences of the associate energy functional Φ_ε enables us to introduce a new functional $\widetilde{\Phi}_\varepsilon$ by virtue of the cut-off technique so that $\widetilde{\Phi}_\varepsilon$ has the same least energy solutions as Φ_ε and can be dealt with more easily under the assumption $\lambda \in (0, \lambda_0]$.

An outline of this paper is as follows: In section 2 we treat the linking argument which gives us a min-max scheme. In section 3, we study the limit equation and introduce the cut-off arguments. Lastly, in section 4, the combination of the results in section 2, 3 proves the Theorem 1.1.

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2 The variational framework

2.1 The functional setting and notations

In the sequel, by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot, \cdot)_2$ the usual L^2-inner product. Let $H_\omega = i\alpha \cdot \nabla - a\beta + \omega$ denote the self-adjoint operator on $L^2 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_\omega) = H^1 \equiv H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is well know that $\sigma(H_\omega) = \sigma_c(H_\omega) = \mathbb{R} \setminus (-a + \omega, a + \omega)$ where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and the continuous spectrum. For $\omega \in (-a, a)$, the space L^2 possesses the orthogonal decomposition:

$$(2.1) \qquad\qquad L^2 = L^+ \oplus L^-, \quad u = u^+ + u^-$$

so that H_ω is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $E := \mathcal{D}(|H_\omega|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$\langle u, v \rangle = \Re(|H_\omega|^{1/2} u, |H_\omega|^{1/2} v)_2$$

and the induced norm $\|u\| = \langle u, u \rangle^{1/2}$, where $|H_\omega|$ and $|H_\omega|^{1/2}$ denote respectively the absolute value of H_ω and the square root of $|H_\omega|$. Since $\sigma(H_\omega) = \mathbb{R} \setminus (-a + \omega, a + \omega)$, one has

$$(2.2) \qquad\qquad (a - |\omega|)|u|_2^2 \leq \|u\|^2 \quad \text{for all } u \in E.$$

Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L^q_{loc} for all $q \in [1, 3)$. It is clear that E possesses the following decomposition

$$(2.3) \qquad\qquad E = E^+ \oplus E^- \quad \text{with} \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot, \cdot)_2$ and $\langle \cdot, \cdot \rangle$ inner products. This decomposition induces also a natural decomposition of L^p, hence there is $d_p > 0$ such that

$$(2.4) \qquad\qquad d_p \left| u^\pm \right|_p^p \leq |u|_p^p \quad \text{for all } u \in E.$$

Let $H^1(\mathbb{R}^3, \mathbb{R})$ be equipped with the equivalent norm

$$\|v\|_{H^1} = \left(\int |\nabla v|^2 + Mv^2 dx \right)^{1/2} \quad \forall v \in H^1(\mathbb{R}^3, \mathbb{R}).$$

Then (1.5) can be reduced to a single equation with a non-local term. Actually, for any $v \in H^1$,

$$(2.5) \qquad
\begin{aligned}
\left| 4\pi\lambda \int (\beta u)u \cdot v\, dx \right| &\leq \left(4\pi\lambda \int |u|^2 |v|\, dx \right) \\
&\leq 4\pi\lambda |u|_{12/5}^2 |v|_6 \\
&\leq 4\pi\lambda S^{-1/2} |u|_{12/5}^2 \|v\|_{H^1},
\end{aligned}$$

where S is the Sobolev embedding constant: $S|v|_6^2 \le \|v\|_{H^1}^2$ for all $v \in H^1$. Hence there exists a unique $V_u \in H^1$ such that

$$(2.6) \qquad \int \nabla V_u \cdot \nabla z + M \cdot V_u z \, dx = 4\pi\lambda \int (\beta u) u \cdot z \, dx$$

for all $z \in H^1$. It follows that V_u satisfies the Schrödinger type equation

$$(2.7) \qquad -\Delta V_u + M \cdot V_u = 4\pi\lambda(\beta u)u$$

and there holds

$$(2.8) \qquad V_u(x) = \lambda \int_{\mathbb{R}^3} \frac{[(\beta u)u](y)}{|x-y|} e^{-M|x-y|} \, dy.$$

Substituting V_u in (1.5), we are led to the equation

$$(2.9) \qquad H_\omega u - \lambda V_u \beta u = W_\varepsilon(x) g(|u|) u.$$

On E we define the functional

$$\Phi_\varepsilon(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Gamma_\lambda(u) - \Psi_\varepsilon(u)$$

for $u = u^+ + u^-$, where

$$\Gamma_\lambda(u) = \frac{\lambda}{4} \int V_u \cdot (\beta u) u \, dx = \frac{\lambda^2}{4} \iint \frac{[(\beta u)u](x)[(\beta u)u](y)}{|x-y|} e^{-M|x-y|} \, dy dx$$

and

$$\Psi_\varepsilon(u) = \int W_\varepsilon(x) G(|u|) dx.$$

2.2 Technical results

In this subsection, we shall introduce some lemmas related to the functional Φ_ε.

Lemma 2.1. *Under the hypotheses (P_0), (G_1)-(G_2), one has $\Phi_\varepsilon \in C^2(E, \mathbb{R})$ and any critical point of Φ_ε is a solution of (1.5).*

Proof. Clearly, $\Psi_\varepsilon \in C^2(E, \mathbb{R})$. It remains to check that $\Gamma_\lambda \in C^2(E, \mathbb{R})$. It suffices to show that, for any $u, v \in E$,

$$(2.10) \qquad |\Gamma_\lambda(u)| \le C_1 \lambda^2 \|u\|^4,$$

$$(2.11) \qquad \left|\Gamma_\lambda'(u)v\right| \le C_2 \lambda^2 \|u\|^3 \|v\|,$$

$$(2.12) \qquad \left|\Gamma_\lambda''(u)[v, v]\right| \le C_3 \lambda^2 \|u\|^2 \|v\|^2.$$

Observe that one has, by using V_u as a test function in (2.7),

$$(2.13) \qquad |V_u|_6 \leq S^{-1/2} \|V_u\|_{H^1} \leq C_1 \lambda \|u\|^2.$$

This, together with the Hölder inequality (with $r = 6, r' = 6/5$), implies (2.10). Note that $\Gamma'_\lambda(u)v = \frac{d}{dt}\Gamma_\lambda(u + tv)\big|_{t=0}$, so

$$
\begin{aligned}
\Gamma'_\lambda(u)v = & \frac{\lambda^2}{2} \Re \iint \frac{e^{-M|x-y|}}{|x-y|} \Big([(\beta u)u](x)[(\beta u)v](y) \\
(2.14) \qquad & + [(\beta u)u](y)[(\beta u)v](x) \Big) dydx \\
= & \lambda \int V_u \cdot \Re(\beta u)v \, dx
\end{aligned}
$$

which, together with the Hölder inequality and (2.13), shows (2.11). Similarly,

$$
\begin{aligned}
\Gamma''_\lambda(u)[v,v] = & 2\lambda^2 \iint \frac{e^{-M|x-y|}}{|x-y|} \Big(\Re[(\beta u)v](x)\Re[(\beta u)v](y) \Big) dxdy \\
& + \lambda \Re \int V_u \cdot (\beta v)v \,,
\end{aligned}
$$

and one gets (2.12).

Now it is a standard to verify that critical points of Φ_ε are solutions of (1.5). $\qquad \square$

We show further the following:

Proposition 2.2. Γ_λ *is non-negative and weakly sequentially lower semi-continuous. Moreover, Γ_λ vanishes only when $(\beta u)u = 0$ a.e. in $\mathbb{R}^3$.*

Proof. Recall that for every $u \in E$, V_u solves (in the weak sense)

$$-\Delta V_u + M V_u = 4\pi\lambda(\beta u)u.$$

Then a standard maximum principle argument shows that

$$(2.15) \qquad \Big(V_u \cdot (\beta u)u \Big)(x) \geq 0, \quad \text{a.e. on } \mathbb{R}^3.$$

Hence (see (2.8))

$$\Gamma_\lambda(u) = \frac{\lambda}{4} \int V_u \cdot (\beta u)u dx \geq 0.$$

Furthermore, suppose $u_n \rightharpoonup u$ in E, then $u_n \to u$ a.e.. Therefore (2.15) and Fatou's lemma yield

$$\Gamma_\lambda(u) \leq \liminf_{n\to\infty} \Gamma_\lambda(u_n)$$

as claimed. $\qquad \square$

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Set, for $r > 0$, $B_r = \{u \in E : \|u\| \leq r\}$, and for $e \in E^+$

$$E_e := E^- \oplus \mathbb{R}^+ e$$

with $\mathbb{R}^+ = [0, +\infty)$. In virtue of the assumptions (G_1)-(G_2), for any $\delta > 0$, there exist $r_\delta > 0$, $c_\delta > 0$ and $c'_\delta > 0$ such that

(2.16)
$$\begin{cases} g(s) < \delta \quad \text{for all } 0 \leq s \leq r_\delta; \\ G(s) \geq c_\delta \, s^\theta - \delta \, s^2 \quad \text{for all } s \geq 0; \\ G(s) \leq \delta \, s^2 + c'_\delta \, s^p \quad \text{for all } s \geq 0 \end{cases}$$

and

(2.17) $$\widehat{G}(s) := \frac{1}{2}g(s)s^2 - G(s) \geq \frac{\theta - 2}{2\theta}g(s)s^2 \geq \frac{\theta - 2}{2}G(s) \geq c_\theta s^\sigma$$

for all $s \geq 0$, where $c_\theta = c_0(\theta - 2)/2$.

Lemma 2.3. *For all $\varepsilon \in (0, 1]$, Φ_ε possess the linking structure:*

1) *There are $r > 0$ and $\tau > 0$, both independent of ε, such that $\Phi_\varepsilon|_{B_r^+} \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \tau$, where*

$$B_r^+ = B_r \cap E^+ = \{u \in E^+ : \|u\| \leq r\},$$

$$S_r^+ = \partial B_r^+ = \{u \in E^+ : \|u\| = r\}.$$

2) *For any $e \in E^+ \setminus \{0\}$, there exist $R = R_e > 0$ and $C = C_e > 0$, both independent of ε, such that, for all $\varepsilon > 0$, there hold $\Phi_\varepsilon(u) < 0$ for all $u \in E_e \setminus B_R$ and $\max \Phi_\varepsilon(E_e) \leq C$.*

Proof. Recall that $|u|_p^p \leq C_p \|u\|^p$ for all $u \in E$ by Sobolev embedding theorem. 1) follows easily because, for $u \in E^+$ and $\delta > 0$ small enough

$$\Phi_\varepsilon(u) = \frac{1}{2} \|u\|^2 - \Gamma_\lambda(u) - \Psi_\varepsilon(u)$$

$$\geq \frac{1}{2} \|u\|^2 - C_1 \lambda^2 \|u\|^4 - |W|_\infty \left(\delta \, |u|_2^2 + c'_\delta \, |u|_p^p \right)$$

with C_1, C_p independent of u and $p > 2$ (see (2.10) and (2.16)).

For checking 2), take $e \in E^+ \setminus \{0\}$. In virtue of (2.4) and (2.16), one gets, for $u = se + v \in E_e$,

(2.18)
$$\Phi_\varepsilon(u) = \frac{1}{2} \|se\|^2 - \frac{1}{2} \|v\|^2 - \Gamma_\lambda(u) - \Psi_\varepsilon(u)$$
$$\leq \frac{1}{2}s^2 \|e\|^2 - \frac{1}{2} \|v\|^2 - c_\delta d_\theta \inf W \cdot s^\theta \, |e|_\theta^\theta$$

proving the conclusion. $\qquad\square$

Recall that a sequence $\{u_n\} \subset E$ is called to be a $(PS)_c$-sequence for functional $\Phi \in C^1(E, \mathbb{R})$ if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, and is called to be $(C)_c$-sequence for Φ if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. It is clear that if $\{u_n\}$ is a $(PS)_c$-sequence with $\{\|u_n\|\}$ bounded then it is also a $(C)_c$-sequence. Below we are going to study $(C)_c$-sequences for Φ_ε but firstly we observe the following

Lemma 2.4. *For all $u \in E$, we have*

$$\left| \frac{V_u}{\lambda \|u\|} \right|_6 \le C|u|_\sigma,$$

where $\sigma > 0$ is the constant in (G_2) and $C > 0$ is depending only on the embedding $H^1(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^6$ and $E \hookrightarrow L^q$ for $\frac{1}{\sigma} + \frac{1}{q} + \frac{1}{6} = 1$.

Proof. Notice that V_u satisfies the equation

$$-\Delta V_u + M V_u = 4\pi \lambda(\beta u)u,$$

hence, using V_u as a test function,

$$\|V_u\|_{H^1}^2 \le 4\pi\lambda \int |V_u| \cdot |u|^2$$

By Hölder's inequality

$$
\begin{aligned}
\|V_u\|_{H^1}^2 &\le 4\pi\lambda |V_u|_6 |u|_\sigma |u|_q \\
&\le 4\pi\lambda \tilde{C} \|V_u\|_{H^1} \cdot \|u\| \cdot |u|_\sigma .
\end{aligned}
$$

And then we infer

$$\left\| \frac{V_u}{\lambda \|u\|} \right\|_{H^1} \le C|u|_\sigma,$$

which yields the conclusion. $\qquad\square$

We now turn to an estimate on boundedness of $(C)_c$-sequences which is the key ingredient in the sequel. Recall that, by (G_1), there exist $r_1 > 0$ and $a_1 > 0$ such that

(2.19) $$g(s) \le \frac{a - |\omega|}{2|W|_\infty} \quad \text{for all } s \le r_1,$$

and, for $s \ge r_1$, $g(s) \le a_1 s^{p-2}$, so $g(s)^{\sigma_0 - 1} \le a_2 s^2$ with

$$\sigma_0 := \frac{p}{p-2} > 3$$

which, jointly with (G_2), yields (see (2.17))

(2.20) $$g(s)^{\sigma_0} \le a_2 g(s) s^2 \le a_3 \widehat{G}(s) \quad \text{for all } s \ge r_1.$$

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Lemma 2.5. *Assume* (P_0), (G_1)-(G_2) *and* $\lambda > 0$, *for every pair of constants* $c_1, c_2 > 0$, *there exists a constant* $\Lambda > 0$, *depending only on* c_1, c_2, λ, *such that for any* $u \in E$ *with*

(2.21) $$|\Phi_\varepsilon(u)| \le c_1 \quad \text{and} \quad \|u\| \cdot \|\Phi_\varepsilon'(u)\| \le c_2,$$

we have

$$\|u\| \le \Lambda.$$

Furthermore, Λ *is a increasing function with respect to* $\lambda > 0$.

Lemma 2.5 has a immediate consequence which implies the boundness of a $(C)_c$-sequence:

Corollary 2.6. *Consider* $\varepsilon \in (0,1]$, *and* $\{u_n^\varepsilon\}$ *is the corresponding* $(C)_{c_\varepsilon}$-*sequence for* Φ_ε. *If there exists* $C > 0$ *such that* $|c_\varepsilon| \le C$ *for all* ε, *then we have (up to a subsequence if necessary)*

$$\|u_n^\varepsilon\| \le \Lambda$$

where Λ *is found in Lemma 2.5 depends on* λ *and the pair* $c_1 = C$ *and* $c_2 = 1$.

Proof of Lemma 2.5. Take $u \in E$ such that (2.21) is satisfied. Without loss of generality we may assume that $\|u\| \ge 1$. The form of Φ_ε and the representation (2.14) $(\Gamma_\lambda'(u)u = 4\Gamma_\lambda(u))$ implies that

(2.22) $$c_1 + c_2 \ge \Phi_\varepsilon(u) - \frac{1}{2}\Phi_\varepsilon'(u)u = \Gamma_\varepsilon(u) + \int W_\varepsilon(x)\widehat{G}(|u|)$$

and

(2.23) $$\begin{aligned} c_2 &\ge \Phi_\varepsilon'(u)(u^+ - u^-) \\ &= \|u\|^2 - \Gamma_\lambda'(u)(u^+ - u^-) \\ &\quad - \Re \int W_\varepsilon(x)g(|u|)u \cdot (u^+ - u^-). \end{aligned}$$

By Lemma 2.2, (2.17) and (2.22), $|u|_\sigma \le C_1$, where C_1 depends only on c_1, c_2. It follows from (2.23) that

$$\|u\|^2 \le c_2 + \Gamma_\lambda'(u)(u^+ - u^-) + \Re \int W_\varepsilon(x)g(|u|)u \cdot (u^+ - u^-).$$

This, together with (2.19) and (2.2), shows

(2.24) $$\frac{1}{2}\|u\|^2 \le c_2 + \Gamma_\lambda'(u)(u^+ - u^-) + \Re \int_{|u| \ge r_1} W_\varepsilon(x)g(|u|)u \cdot (u^+ - u^-).$$

Recall that (G_1) and (G_2) imply $2 < \sigma \le p$. Setting $t = \frac{p\sigma}{2\sigma - p}$, one sees

$$2 < t < p, \quad \frac{1}{\sigma_0} + \frac{1}{\sigma} + \frac{1}{t} = 1.$$

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By Hölder inequality, the fact $\Gamma_\lambda(u) \geq 0$, (2.20), (2.22) and the embedding of E to L^t, we have

$$\int_{|u| \geq r_1} W_\varepsilon(x)\, g(|u|)|u| \cdot |u^+ - u^-|$$

(2.25)
$$\leq |W|_\infty \Big(\int_{|u| \geq r_1} g(|u|)^{\sigma_0} \Big)^{1/\sigma_0} \Big(\int |u|^\sigma \Big)^{1/\sigma} \Big(|u^+ - u^-|^t \Big)^{1/t}$$

$$\leq C_2 \|u\|$$

with $C_2 > 0$ depends only on c_1, c_2.

Let $q = \frac{6\sigma}{5\sigma - 6}$. Then $2 < q < 3$ and $\frac{1}{\sigma} + \frac{1}{q} + \frac{1}{6} = 1$. Set

$$\zeta = \begin{cases} 0 & \text{if } q = \sigma; \\[2mm] \dfrac{2(\sigma - q)}{q(\sigma - 2)} & \text{if } q < \sigma; \\[3mm] \dfrac{3(q - \sigma)}{q(3 - \sigma)} & \text{if } q > \sigma; \end{cases}$$

we deduce that $\zeta < 1$ and

$$|u|_q \leq \begin{cases} |u|_2^\zeta \cdot |u|_\sigma^{1-\zeta} & \text{if } 2 < q \leq \sigma \\[1mm] |u|_3^\zeta \cdot |u|_\sigma^{1-\zeta} & \text{if } \sigma < q < 3. \end{cases}$$

By virtue of the Hölder inequality, Lemma 2.2 and the embedding of E to L^2 and L^3, we obtain

$$\left| \lambda \Re \int V_u \cdot (\beta u)(u^+ - u^-) \right|$$

$$\leq \lambda \|u\| \left| \Re \int \frac{V_u}{\|u\|} (\beta u) \cdot (u^+ - u^-) \right|$$

$$\leq \lambda^2 \|u\| \left| \frac{V_u}{\lambda \|u\|} \right|_6 |u|_\sigma \cdot |u^+ - u^-|_q$$

$$\leq \lambda^2 C_3 \|u\| \cdot |u|_q \leq \lambda^2 C_4 \|u\|^{1+\zeta}$$

with $C_4 > 0$ depends only on the embedding $E \hookrightarrow L^q$. This, together with the representation of (2.14), implies that

(2.26)
$$|\Gamma_\lambda'(u)(u^+ - u^-)| \leq \lambda^2 C_4 \|u\|^{1+\zeta}.$$

Now the combination of (2.24), (2.25) and (2.26) shows that

(2.27)
$$\|u\|^2 \leq M_1 \|u\| + \lambda^2 M_2 \|u\|^{1+\zeta}$$

with M_1 and M_2 dependent only on the constants c_1, c_2. Therefore, either $\|u\| \leq 1$ or there is $\Lambda \geq 1$ dependents only on c_1, c_2, λ such that

$$\|u\| \leq \Lambda$$

as desired. Moreover, (2.27) implies Λ is increasing in λ. □

Finally, for later aims we define the operator $\mathcal{V} : E \to H^1(\mathbb{R}^3, \mathbb{R})$ by $\mathcal{V}(u) = V_u$. We have

Lemma 2.7. (1) $\mathcal{V}$ *maps bounded sets into bounded sets;*

(2) $\mathcal{V}$ *is continuous;*

Proof. Clearly, (1) is a straight consequence of (2.13). (2) follows easily because, for $u, v \in E$, one sees that $V_u - V_v$ satisfies

$$-\Delta(V_u - V_v) + M(V_u - V_v) = 4\pi\lambda[(\beta u)u - (\beta v)v].$$

Hence

$$\begin{aligned}
\|V_u - V_v\|_{H^1} &\leq \lambda C \big|(\beta u)u - (\beta v)v\big|_{6/5} \\
&\leq \lambda C \Big(|u - v|_{12/5}\, |u|_{12/5} + |u - v|_{12/5}\, |v|_{12/5} \Big) \\
&\leq \lambda \tilde{C} \big(\|u - v\| \cdot \|u\| + \|u - v\| \cdot \|v\| \big).
\end{aligned}$$

and this implies the desired conclusion. □

3 Preliminary results

We are interested in describing the concentration phenomena of the least energy solutions to the semi-classical model (1.5). Throughout this section we will collect properties of the energy functionals of the Dirac-Klein-Gordon systems (including the estimates of the least energy). Instead of dealing directly with the nonlocal term Γ_λ, it seems simpler to consider a modified problem (see subsection 3.2). For reasons that will be apparent later, we treat our model in the case λ is not chosen large, that is $\lambda \in (0, \lambda_0]$ for some $\lambda_0 > 0$ will be chosen later on.

3.1 The limit equation

In order to prove our main result, we will make use of the limit equation. For any $\mu > 0$, consider the equation

$$(3.1) \qquad \begin{cases} i\alpha \cdot \nabla u - a\beta u + \omega u - \lambda V\beta u = \mu g(|u|)u, \\ -\Delta V + M \cdot V = 4\pi\lambda(\beta u)u. \end{cases}$$

Its solutions are critical points of the functional

$$\mathscr{T}_\mu(u) := \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \Gamma_\lambda(u) - \mu \int G(|u|)$$

$$= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \Gamma_\lambda(u) - \mathscr{G}_\mu(u)$$

defined for $u = u^+ + u^- \in E = E^+ \oplus E^-$. Denote the critical set and the least energy of $\mathscr{T}_\mu$ as follows

$$\mathscr{K}_\mu := \{u \in E : \ \mathscr{T}_\mu'(u) = 0\},$$
$$\gamma_\mu := \inf\{\mathscr{T}_\mu(u) : \ u \in \mathscr{K}_\mu \setminus \{0\}\}.$$

In order to find critical points of $\mathscr{T}_\mu$, we will use the following abstract theorem which is taken from [5, 11].

Let E be a Banach space with direct sum decomposition $E = X \oplus Y$, $u = x + y$ and corresponding projections P_X, P_Y onto X, Y, respectively. For a functional $\Phi \in C^1(E, \mathbb{R})$ we write $\Phi_a = \{u \in E : \Phi(u) \geq a\}$.

Now we assume that X is separable and reflexive, and we fix a countable dense subset $\mathscr{S} \subset X^*$. For each $s \in \mathscr{S}$ there is a semi-norm on E defined by

$$p_s : E \to \mathbb{R}, \quad p_s(u) = |s(x)| + \|y\| \quad \text{for } u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_\mathscr{S}$ the induced topology. Let w^* denote the weak*-topology on E. Suppose:

(Φ_0) There exists $\xi > 0$ such that $\|u\| < \xi\|P_Y u\|$ for all $u \in \Phi_0$.

(Φ_1) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_\mathscr{S}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_\mathscr{S}) \to (E^*, w^*)$ is continuous.

(Φ_2) There exists $\rho > 0$ with $\kappa := \inf \Phi(S_\rho Y) > 0$ where $S_\rho Y := \{u \in Y : \|u\| = \rho\}$.

The following theorem is a special case of [5] Theorem 3.4 (see also [11] Theorem 4.3).

Theorem 3.1. *Let $(\Phi_0) - (\Phi_2)$ be satisfied and suppose there are $R > \rho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi(\partial Q) \leq \kappa$ where $Q = \{u = x + te : x \in X, t \geq 0, \|u\| < R\}$. Then Φ has a $(C)_c$-sequence with $\kappa \leq c \leq \sup \Phi(Q)$.*

The following lemma is useful to verify (Φ_1) (see [5] or [11]).

Lemma 3.2. *Suppose $\Phi \in C^1(E, \mathbb{R})$ is of the form*

$$\Phi(u) = \frac{1}{2}\big(\|y\|^2 - \|x\|^2\big) - \Psi(u) \quad \text{for } u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ is bounded from below;

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ is sequentially lower semi-continuous, that is, $u_n \rightharpoonup u$ in E implies $\Psi(u) \leq \liminf \Psi(u_n)$;

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, w^*)$ is sequentially continuous.

(iv) $\nu : E \to \mathbb{R}$, $\nu(u) = \|u\|^2$, is C^1 and $\nu' : (E, \mathcal{T}_w) \to (E^*, w^*)$ is sequentially continuous.

Then Φ satisfies (Φ_1).

Next, we present the existence result for the limit equation (3.1).

Lemma 3.3. *Let λ be a positive constant, for each $\mu > 0$, we have*

1. $\mathscr{K}_\mu \neq \emptyset$ and $\gamma_\mu > 0$,

2. γ_μ is attained.

Proof. Invoking Lemma 2.2, we see that (Φ_0) is satisfied. With $X = E^-$ and $Y = E^+$ the condition (Φ_0) holds by Lemma 2.2 and Lemma 3.2. Together with the linking structure (see Lemma 2.3) we have all the assumptions of Theorem 3.1 verified. Therefore, there exists a sequence $\{u_m\}$ satisfying $\mathscr{T}_\mu(u_m) \to c > 0$ and $(1 + \|u_m\|)\mathscr{T}'_\mu(u_m) \to 0$ as $m \to \infty$. Using the same arguments in proving Lemma 2.5, we get $\{u_m\}$ is bounded. Now by the classical concentration compactness principle (cf. [24]) and the translation-invariance of $\mathscr{T}_\mu$, we infer there is $u \neq 0$ such that $\mathscr{T}'_\mu(u) = 0$.

If $u \in \mathscr{K}_\mu$, one has

$$(3.2) \qquad \mathscr{T}_\mu(u) = \mathscr{T}_\mu(u) - \frac{1}{2}\mathscr{T}'_\mu(u)u = \Gamma_\lambda(u) + \mu \int \widehat{G}(u) \geq 0.$$

For proving $\gamma_\mu > 0$, assume by contradiction that $\gamma_\mu = 0$. Let $u_j \in \mathscr{K}_\mu \setminus \{0\}$ such that $\mathscr{T}_\mu(u_j) \to 0$. It is obvious that $\{u_j\}$ is bounded. Furthermore, by (2.17) and (3.2), we deduce $u_j \to 0$ in L^σ as $j \to \infty$. On the other hand, by noting that $0 = \mathscr{T}'_\mu(u_j)(u_j^+ - u_j^-)$, (2.4) and Lemma 2.4 imply

$$\|u_j\|^2 = \Gamma'_\lambda(u_j)(u_j^+ - u_j^-) + \mu \int g(u_j)(u_j^+ - u_j^-)$$

$$\leq \lambda^2 C_1 \|u_j\|^3 \cdot |u_j|_\sigma + \mu \int g(u_j)(u_j^+ - u_j^-).$$

By (2.20) and Hölder's inequality, one sees

$$\frac{1}{2}\|u_j\|^2 \leq \lambda^2 C_1 \|u_j\|^3 \cdot |u_j|_\sigma + C_2\mu \left(\int g(|u_j|)^{\sigma_0} \right)^{1/\sigma_0} |u_j|_p^2$$

$$\leq \lambda^2 C_1 \|u_j\|^3 \cdot |u_j|_\sigma + C_3\mu \left(\mathscr{T}_\mu(u_j) \right)^{1/\sigma_0} \|u_j\|^2.$$

Hence $\frac{1}{2} \leq o(1) + o(1)$, a contradiction.

Lastly, again, by using the concentration compactness principle, we check easily that γ_μ is attained, ending the proof. $\square$

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3.2 A modification for the nonlocal term

We find our current research is more delicate, since the solutions we look for are at the least energy level and Γ_λ is not convex on E (even for u with $\|u\|$ large). By cutting off the nonlocal terms, we are able to find a critical point via an appropriate min-max scheme. The critical point will eventually be shown to be a least energy solution to our model.

Next we introduce the modified problem by choosing a cut-off function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\mathscr{F}_\lambda(u) := \eta(\|u\|^2)\Gamma_\lambda(u)$ vanishes for $\|u\|$ large.

By virtue of (P_0), set $b = \inf W(x) > 0$, let us first consider the autonomous systems for $\mu \geq b$

$$
\begin{cases}
i\alpha \cdot \nabla u - a\beta u + \omega u - \lambda V \beta u = \mu g(|u|)u, \\
-\Delta V + M \cdot V = 4\pi\lambda(\beta u)u.
\end{cases}
$$

Following Lemma 3.3, $\gamma_\mu > 0$ (the least energy) is attained. Now fix $\Lambda > 0$ to be the constant (independent of $\varepsilon > 0$) found in Lemma 2.5 associated to $\lambda > 0$ and the pair of the constant $c_1 = C_{e_0}$ and $c_2 = 1$, where C_{e_0} (independent of λ and μ) is the constant in Lemma 2.3 with $e_0 \in E^+ \setminus \{0\}$ being fixed.

It is obvious that $\gamma_\mu \leq C_{e_0}$. Denote $T = (\Lambda + 1)^2$ and choose $\eta : [0, +\infty) \to [0, 1]$ be a smooth function with $\eta(t) = 1$ if $0 \leq t \leq T$, $\eta(t) = 0$ if $t \geq T + 1$, $\max|\eta'(t)| \leq 2$ and $\max|\eta''(t)| \leq 2$. Define $\mathscr{F}_\lambda : E \to \mathbb{R}$ as $\mathscr{F}_\lambda(u) = \eta(\|u\|^2)\Gamma_\lambda(u)$. Then we have $\mathscr{F}_\lambda \in C^2(E, \mathbb{R})$ and $\mathscr{F}_\lambda$ vanishes for all u with $\|u\| \geq \sqrt{T + 1}$.

Consider the modified functionals

$$
\widetilde{\mathscr{T}_\mu}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \mathscr{F}_\lambda(u) - \mathscr{G}_\mu(u),
$$

and

$$
\widetilde{\Phi}_\varepsilon(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \mathscr{F}_\lambda(u) - \Psi_\varepsilon(u).
$$

By definition, $\widetilde{\mathscr{T}_\mu}\big|_{B_T} = \mathscr{T}_\mu$ and $\widetilde{\Phi}_\varepsilon\big|_{B_T} = \Phi_\varepsilon$ where $B_T := \{u \in E : \|u\| \leq \sqrt{T}\}$. And it's easy to see that $0 \leq \mathscr{F}_\lambda(u) \leq \Gamma_\lambda(u)$ and

$$
\big|\mathscr{F}_\lambda'(u)v\big| \leq \big|2\eta'(\|u\|^2)\Gamma_\lambda(u)\langle u, v\rangle\big| + \big|\Gamma_\lambda'(u)v\big|
$$

for $u, v \in E$.

Similarly to Lemma 2.5, we have the following boundedness lemma (with Λ being taken as above):

Lemma 3.4. *Assume (G_1)-(G_2) and (P_0). There exists $\lambda_1 > 0$ such that, for each $\lambda \in (0, \lambda_1]$, if $u \in E$ satisfies*

$$
(3.3) \qquad 0 \leq \widetilde{\Phi}_\varepsilon(u) \leq C_{e_0} \quad \text{and} \quad \|u\| \cdot \big\|\widetilde{\Phi}_\varepsilon'\big\| \leq 1,
$$

then we have $\|u\| \leq \Lambda + 1$, and consequently $\widetilde{\Phi}_\varepsilon(u) = \Phi_\varepsilon(u)$.

In particular, replace $\widetilde{\Phi}_\varepsilon$ with $\widetilde{\mathscr{T}_\mu}$, we have $\widetilde{\mathscr{T}_\mu}$ shares the same ground state solution with $\mathscr{T}_\mu$.

Proof. We repeat the arguments of Lemma 2.5. Let u satisfy (3.3). If $\|u\|^2 \geq T + 1$ then $\mathscr{F}_\lambda(u) = 0$ so, as proved in Lemma 2.5, one changes (2.27) by $\|u\|^2 \leq M_1\|u\|$ and gets $\|u\| \leq \Lambda$, a contradiction. Thus we assume that $\|u\|^2 \leq T + 1$. Then, using (2.10), $|\eta'(\|u\|^2)\|u\|^2\Gamma_\lambda(u)| \leq \lambda^2 d_\lambda^{(1)}$ (here and in the following, by $d_\lambda^{(j)}$ we denote positive constants depending only on λ and $d_\lambda^{(j)}$ is increasing with respect to λ). Similar to (2.22),

$$C_{e_0} + 1 \geq \left(\eta(\|u\|^2) + 2\eta'(\|u\|^2)\|u\|^2\right)\Gamma_\lambda(u) + \int W_\varepsilon(x)\widehat{G}(|u|)$$

which yields

$$C_{e_0} + 1 + \lambda^2 d_\lambda^{(1)} > \eta(\|u\|^2)\Gamma_\lambda(u) + \int W_\varepsilon(x)\widehat{G}(|u|),$$

consequently $|u|_\sigma \leq d_\lambda^{(2)}$. Similarly to (2.24) we get that

$$\frac{1}{2}\|u\|^2 \leq \lambda^2 d_\lambda^{(3)} + \eta(\|u\|^2)\Gamma'_\lambda(u)(u^+ - u^-)$$
$$+ \Re \int_{|u| \geq r_1} W_\varepsilon(x)g(|u|)u \cdot \overline{u^+ - u^-}$$

which, together with (2.25) and (2.26), implies either $\|u\| \leq 1$ or as (2.27)

$$\|u\|^2 \leq \lambda^2 d_\lambda^{(4)} + M_1\|u\| + M_2\|u\|^{1+\zeta},$$

thus

$$\|u\| \leq \lambda^2 d_\lambda^{(5)} + \Lambda.$$

By monotonicity of $d_\lambda^{(j)}$, we see that, for $\lambda_1 > 0$ being suitably chosen, let $\lambda \in (0, \lambda_1]$ then $\|u\| \leq \Lambda + 1$. The proof is complete. $\square$

3.3 Estimates on the least energy

Under Lemma 3.4, instead of study directly on Φ_ε and $\mathscr{T}_\mu$, we turn to investigate the modified functionals, that is, $\widetilde{\Phi}_\varepsilon$ and $\widetilde{\mathscr{T}_\mu}$ respectively. This will give more information on the least energy level and more descriptions on the min-max scheme.

Firstly, following the definitions of the modified functionals, an easy observation shows:

Proposition 3.5. $\widetilde{\Phi}_\varepsilon$ *and* $\widetilde{\mathscr{T}_\mu}$ *possess the linking structure proved in Lemma 2.3, and the constants found in Lemma 2.3 are independent of the choice of* $\widetilde{\Phi}_\varepsilon$, Φ_ε, $\widetilde{\mathscr{T}_\mu}$ *or* $\mathscr{T}_\mu$, *where* $\mu \geq b$.

Now let us define (see [5, 30])

$$(3.4) \qquad c_\varepsilon := \inf_{z \in E^+ \backslash \{0\}} \max_{u \in E_z} \widetilde{\Phi}_\varepsilon(u) \quad \text{and} \quad \tilde{\gamma}_\mu := \inf_{z \in E^+ \backslash \{0\}} \max_{u \in E_z} \widetilde{\mathscr{T}}_\mu(u)$$

As a consequence of Proposition 3.5 and Lemma 3.4 we have

Lemma 3.6. $c_\varepsilon, \tilde{\gamma}_\mu \in [\tau, C_{e_0}]$. *Moreover, consider* $\mu \geq b$, *if* c_ε *and* $\tilde{\gamma}_\mu$ *are critical values for* $\widetilde{\Phi}_\varepsilon$ *and* $\widetilde{\mathscr{T}}_\mu$, *then they are also critical values for* Φ_ε *and* $\mathscr{T}_\mu$ *respectively.*

For a specific description, let us introduce the following notations: Consider $\mu \geq b$, define

$$\mathcal{I} = \begin{cases} \widetilde{\Phi}_\varepsilon & \text{for the nonautonomous system,} \\ \widetilde{\mathscr{T}}_\varepsilon & \text{for the autonomous system.} \end{cases}$$

Following Ackermann [1] (also see [12, 13, 15]), for any fixed $u \in E^+$, let $\varphi_u : E^- \to \mathbb{R}$ defined by $\varphi_u(v) = \mathcal{I}(u+v)$. We have, for any $v, w \in E^-$,

$$\varphi_u''(v)[w, w] \leq -\|w\|^2 - \mathscr{F}_\lambda''(u+v)[w, w].$$

At this point, a direct computation shows

$$\begin{aligned} & \mathscr{F}_\lambda(u+v)''[w, w] \\ &= \left(4\eta''(\|u+v\|^2) \, |\langle u+v, w \rangle|^2 + 2\eta'(\|u+v\|^2) \, \|w\|^2 \right) \Gamma_\lambda(u+v) \\ & \quad + 4\eta'(\|u+v\|^2) \, \langle u+v, w \rangle \, \Gamma_\lambda'(u+v)w + \eta(\|u+v\|^2)\Gamma_\lambda''(u+v)[w, w]. \end{aligned}$$

Combining (2.10)-(2.12) yields

$$\left| \mathscr{F}_\lambda''(u+v)[w, w] \right| \leq \lambda^2 d_\lambda \|w\|^2 \leq \frac{1}{2}\|w\|^2$$

for $\lambda \leq \lambda_2$, where λ_2 is suitably chosen (here d_λ is a positive constant depending monotonically only on λ). Hence, by setting $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, for each $\lambda \in (0, \lambda_0]$ we deduce

$$\varphi_u''(v)[w, w] \leq -\frac{1}{2}\|w\|.$$

Additionally, we find

$$\varphi_u(v) \leq \frac{1}{2}\big(\|u\|^2 - \|v\|^2\big).$$

Therefore, there exists a unique $\xi : E^+ \to E^-$ such that

$$\mathcal{I}(u + \xi(u)) = \max_{v \in E^-} \mathcal{I}(u+v).$$

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Here we used the expressions

$$\xi(u) = \begin{cases} h_\varepsilon(u) & \text{defined for the nonautonomous system,} \\ \mathscr{J}_\mu(u) & \text{defined for the autonomous system.} \end{cases}$$

In the sequel, we fix λ in the interval $(0, \lambda_0]$. Next, setting $I_\varepsilon, J_\mu : E^+ \to \mathbb{R}$ by

$$I_\varepsilon(u) = \widetilde{\Phi}_\varepsilon(u + h_\varepsilon(u)),$$
$$J_\mu(u) = \widetilde{\mathscr{T}_\mu}(u + \mathscr{J}_\mu(u)),$$

and

$$\mathscr{N}_\varepsilon = \{u \in E^+ \setminus \{0\} : \ I'_\varepsilon(u)u = 0\},$$
$$\mathscr{M}_\mu = \{u \in E^+ \setminus \{0\} : \ J'_\mu(u)u = 0\}.$$

Denote by

$$\mathcal{J}(u) = \begin{cases} I_\varepsilon(u) & \text{for the nonautonomous system,} \\ J_\mu(u) & \text{for the autonomous system,} \end{cases}$$

and

$$\mathcal{M} = \begin{cases} \mathscr{N}_\varepsilon & \text{for the nonautonomous system}, \\ \mathscr{M}_\mu & \text{for the autonomous system}. \end{cases}$$

Plainly, critical points of $\mathcal{J}$ and $\mathcal{I}$ are in one-to-one correspondence via the injective map $u \mapsto u + \xi(u)$ from E^+ into E.

Lemma 3.7. *For any $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $tu \in \mathcal{M}$.*

Proof. See [1, 15]. □

To give more information on the min-max levels defined in (3.4), we set

$$d = \begin{cases} c_\varepsilon & \text{for the nonautonomous system}, \\ \tilde{\gamma}_\mu & \text{for the autonomous system}. \end{cases}$$

Proposition 3.8. *There holds:*

1. *$d = \inf_{u \in \mathcal{M}} \mathcal{J}(u)$.*

2. *For $\mu \geq b$, $\tilde{\gamma}_\mu$ is the least energy for $\widetilde{\mathscr{T}_\mu}$ and, by invoking Lemma 3.4, $\tilde{\gamma}_\mu = \gamma_\mu$.*

3. *Let $u \in \mathscr{M}_\mu$ be such that $J_\mu(u) = \tilde{\gamma}_\mu$ and set $E_u = E^- \oplus \mathbb{R}^+ u$. Then*

$$\max_{w \in E_u} \widetilde{\mathscr{T}_\mu}(w) = J_\mu(u).$$

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4. If $\mu_2 > \mu_1 \geq b$, then $\tilde{\gamma}_{\mu_1} > \tilde{\gamma}_{\mu_2}$.

Proof. Denoting $\hat{d} = \inf_{u \in \mathcal{M}} \mathcal{J}(u)$, given $e \in E^+$, if $u = v + se \in E_e$ with $\mathcal{J}(u) = \max_{z \in E_e} \mathcal{I}(z)$ then the restriction $\mathcal{I}|_{E_e}$ of $\mathcal{I}$ on E_e satisfies $(\mathcal{I}|_{E_e})'(u) = 0$ which implies $v = \xi(se)$ and $\mathcal{I}'(se)(se) = 0$, i.e. $se \in \mathcal{M}$. Thus $\hat{d} \leq d$. While, on the other hand, if $w \in \mathcal{M}$ then $(\mathcal{I}|_{E_w})'(w + \xi(w)) = 0$, hence, $d \leq \max_{u \in E_w} \mathcal{I}(u) = \mathcal{J}(w)$. Thus $\hat{d} \geq d$. It follow that $d = \hat{d}$. Since it is standard to see that, for the autonomous system, $\inf_{u \in \mathcal{M}_\mu} J_\mu(u)$ characterize the least energy, we infer that $\gamma_\mu = \tilde{\gamma}_\mu$. To prove 3, we note that $u + \mathscr{J}_\mu(u) \in E_u$ and

$$J_\mu(u) = \widetilde{\mathscr{T}_\mu}(u + \mathscr{J}_\mu(u)) \leq \max_{w \in E_u} \widetilde{\mathscr{T}_\mu}(w),$$

moreover, since $u \in \mathcal{M}_\mu$,

$$\max_{w \in E_u} \widetilde{\mathscr{T}_\mu}(w) \leq \max_{s \geq 0} \widetilde{\mathscr{T}_\mu}(su + \mathscr{J}_\mu(su)) \leq \max_{s \geq 0} J_\mu(su) = J_\mu(u).$$

Therefore, $\max_{w \in E_u} \widetilde{\mathscr{T}_\mu}(w) = J_\mu(u)$. Lastly to get 4, let u_1 be the ground state solution for $\widetilde{\mathscr{T}_{\mu_1}}$ and set $e = u_1^+$. Then

$$\tilde{\gamma}_{\mu_1} = \widetilde{\mathscr{T}_{\mu_1}}(u_1) = \max_{w \in E_e} \widetilde{\mathscr{T}_{\mu_1}}(w).$$

Suppose $u_2 \in E_e$ be such that $\widetilde{\mathscr{T}_{\mu_2}}(u_2) = \max_{w \in E_e} \widetilde{\mathscr{T}_{\mu_2}}(w)$. We deduce that

$$\tilde{\gamma}_{\mu_1} = \widetilde{\mathscr{T}_{\mu_1}}(u_1) \geq \widetilde{\mathscr{T}_{\mu_1}}(u_2) = \widetilde{\mathscr{T}_{\mu_2}}(u_2) + (\mu_2 - \mu_1) \int G(|u_2|)$$

$$\geq \tilde{\gamma}_{\mu_2} + (\mu_2 - \mu_1) \int G(|u_2|).$$

This ends the proof. $\qquad\square$

Lemma 3.9. *For any $e \in E^+ \setminus \{0\}$, there is $T_e > 0$ independent the choice of $\widetilde{\Phi}_\varepsilon$ or $\widetilde{\mathscr{T}_\mu}$ such that $t_e \leq T_e$ for $t_e > 0$ satisfying $t_e e \in \mathcal{M}$.*

Proof. Since $\mathcal{J}'(t_e e)(t_e e) = 0$, one get

$$\mathcal{I}(t_e e + \xi(t_e e)) = \max_{w \in E_e} \mathcal{I}(w) \geq \tau.$$

This, together with Proposition 3.5 (the linking structure), shows the assertion. $\qquad\square$

3.4 Some auxiliary results

Now using the notations introduced above, we are going to show some auxiliary results that will make our arguments more transparent. First of all, to describe the nonlinearities, we set

$$\mathcal{N}(u) = \begin{cases} \Psi_\varepsilon(u) & \text{for the nonautonomous system}, \\ \mathscr{G}_\mu(u) & \text{for the autonomous system}. \end{cases}$$

For any $u \in E^+$ and $v \in E^-$, setting $z = v - \xi(u)$ and $l(t) = \mathcal{I}(u + \xi(u) + tz)$, one has $l(1) = \mathcal{I}(u + v)$, $l(0) = \mathcal{I}(u + \xi(u))$ and $l'(0) = 0$. Thus $l(1) - l(0) = \int_0^1 (1-t)l''(t)dt$. This implies that

$$\mathcal{I}(u + v) - \mathcal{I}(u + \xi(u))$$
$$= \int_0^1 (1-t)\mathcal{I}''(u + \xi(u) + tz)[z, z]dt$$
$$= -\int_0^1 (1-t)\|z\|^2 dt - \int_0^1 (1-t)\Big[\mathscr{F}_\lambda''(u + \xi(u) + tz)[z, z]$$
$$+ \mathcal{N}''(u + \xi(u) + tz)[z, z]\Big]dt,$$

and hence

(3.5)
$$\int_0^1 (1-t)\Big[\mathscr{F}_\lambda''(u + \xi(u) + tz)[z, z] + \mathcal{N}''(u + \xi(u) + tz)[z, z]\Big]dt$$
$$+ \frac{1}{2}\|z\|^2 = \mathcal{I}(u + \xi(u)) - \mathcal{I}(u + v).$$

Remark 3.10. Recall that, for $\lambda \in (0, \lambda_0]$ being a positive constant, there holds
$$\big|\mathscr{F}_\lambda''(u + \xi(u) + tz)[z, z]\big| \le \frac{1}{2}\|z\|^2.$$

From (3.5), we deduce that, for the autonomous system,

(3.6)
$$\widetilde{\mathscr{T}_\mu}(u + \mathscr{J}_\mu(u)) - \widetilde{\mathscr{T}_\mu}(u + v)$$
$$\ge \frac{1}{4}\|z\|^2 + \int_0^1 (1-t)\mathscr{G}_\mu''(u + \xi(u) + tz)[z, z]dt.$$

Next we estimate the regularity of the critical points of $\widetilde{\Phi}_\varepsilon$. Let $\mathscr{K}_\varepsilon := \{u \in E : \widetilde{\Phi}_\varepsilon'(u) = 0\}$ be the critical set of $\widetilde{\Phi}_\varepsilon$. It is easy to see that if $\mathscr{K}_\varepsilon \setminus \{0\} \ne \emptyset$ then $c_\varepsilon = \inf\big\{\widetilde{\Phi}_\varepsilon(u) : u \in \mathscr{K}_\varepsilon \setminus \{0\}\big\}$ (see an argument of [15]). Using the same iterative argument of [18] one obtains easily the following

Lemma 3.11. *Consider* $\lambda > 0$ *being a constant, if* $u \in \mathscr{K}_\varepsilon$ *with* $|\widetilde{\Phi}_\varepsilon(u)| \le C$, *then, for any* $q \in [2, +\infty)$, $u \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$ *with* $\|u\|_{W^{1,q}} \le \Lambda_q$ *where* Λ_q *depends only on* C *and* q.

Proof. See [18]. We outline the proof as follows. Firstly, from (2.9), we write

$$u = H_\omega^{-1}\Big(\lambda V_u \beta u + W_\varepsilon(x) g(|u|) u\Big).$$

Now let $\rho : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\rho(s) = 1$ if $s \in [0, 1]$ and $\rho(s) = 0$ if $s \in [2, \infty)$. Then we have

$$\begin{aligned} g(s) &:= g_1(s) + g_2(s) \\ &= \rho(s) g(s) + (1 - \rho(s)) g(s). \end{aligned}$$

Consequently, $u = u_1 + u_2 + u_3$ with

$$\begin{aligned} u_1 &= H_\omega^{-1}\big(W_\varepsilon \cdot g_1(|u|) u\big), \\ u_2 &= \lambda H_\omega^{-1}\big(V_u \beta u\big), \\ u_3 &= H_0^{-1}\big(W_\varepsilon \cdot g_2(|u|) u\big). \end{aligned}$$

Next we remark that, by Hölder's inequality, for $q \geq 2$

$$\big|V_u \beta u\big|_s \leq \big|V_u\big|_6 \cdot |u|_q$$

with $\frac{1}{s} = \frac{1}{6} + \frac{1}{q}$ and, jointly with (2.20),

$$\big|W_\varepsilon \cdot g_2(|u|) u\big|_t \leq C_1 |W|_\infty |u|_{t(p-1)}^{p-1},$$

where $C_1 > 0$ is a constant. Hence, we obtain

$$u_1 \in W^{1,2} \cap W^{1,3}, \ u_2 \in W^{1,s}, \ u_3 \in W^{1,t}.$$

Then, denoting $s^* = \frac{3s}{3-s}$ and $t^* = \frac{3t}{3-t}$, one sees $u \in W^{1,q}$ with $q = \min\{s^*, t^*\}$.

Starting with $q = 2$, a standard bootstrap argument shows that $u \in \cap_{q \geq 2} L^q$, $u_1 \in \cap_{q \geq 2} W^{1,q}$, $u_2 \in \cap_{6 > q \geq 2} W^{1,q}$ and $u_3 \in \cap_{q \geq 2} W^{1,q}$.

By Sobolev embedding theorems, $u \in C^{0,\gamma}$ for some $\gamma \in (0, 1)$. This, together with elliptic regularity (see [22]), shows $V_u \in W^{2,2}_{loc}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and

$$\|V_u\|_{W^{2,2}(B_1(x))} \leq C_2\Big(\lambda |u|^2_{L^4(B_2(x))} + \|V_u\|_{H^1(B_2(x))}\Big)$$

for all $x \in \mathbb{R}^3$, with C_2 independent of x and ε, where $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$ for $r > 0$. Since $W^{2,2}(B_1(x)) \hookrightarrow C^{0,\delta}(B_1(x))$, $\delta \in (0, \frac{1}{2})$, we have

$$(3.7) \qquad \|V_u\|_{C^{0,\delta}(B_1(x))} \leq C_3\Big(\lambda |u|^2_{L^4(B_2(x))} + \|V_u\|_{H^1(B_2(x))}\Big)$$

for all $x \in \mathbb{R}^3$ with C_3 independent of x and ε. Consequently $V_u \in L^\infty$, and this yields

$$|V_u \beta u|_s \leq |V_u|_\infty |u|_s.$$

Thus $u_2 \in \cap_{q \geq 2} W^{1,q}$, and combining with $u_1, u_3 \in \cap_{q \geq 2} W^{1,q}$ the conclusion is obtained. $\qquad\square$

Remark 3.12. Let $\mathscr{L}_\varepsilon$ denote the set of all least energy solutions of $\widetilde{\Phi}_\varepsilon$. If $u \in \mathscr{L}_\varepsilon$, then $\widetilde{\Phi}_\varepsilon(u) = c_\varepsilon \leq C_{e_0}$. Recall that $\mathscr{L}_\varepsilon$ is bounded in E with upper bound Λ independent of ε. Therefore, as a consequence of Lemma 3.11 we see that, for each $q \in [2, +\infty)$ there is $C_q > 0$ independent of ε such that

$$(3.8) \qquad \|u\|_{W^{1,q}} \leq C_q \quad \text{for all } u \in \mathscr{L}_\varepsilon.$$

This, together with the Sobolev embedding theorem, implies that there is $C_\infty > 0$ independent of ε with

$$(3.9) \qquad \|u\|_\infty \leq C_\infty \quad \text{for all } u \in \mathscr{L}_\varepsilon.$$

4 Proof of the main result

Throughout this section we assume $\omega \in (-a, a)$, (P_0) and (G_1)-(G_2) are satisfied. We also suppose, without loss of generality, that $0 \in \mathscr{C}$. The proof of the main theorem will be achieved in three parts: *Existence, Concentration, and Exponential decay.*

Part 1. Existence

Keeping the notation of Section 3 we now turn to the existence result of the main theorem. Its proof is carried out in three lemmas. The modified problem gives us an access to Lemma 4.1, which is the key ingredient for Lemma 4.2.

Recall that $\tilde{\gamma}_m$ denotes the least energy of $\widetilde{\mathscr{T}_m}$ (see the subsection 3.2), where $\mu = m := \max_{x \in \mathbb{R}^3} W(x)$, and J_m denotes the associated reduction functional on E^+. We remark that, since $0 \in \mathscr{C}$, $W_\varepsilon(x) \to W(0) = m$ uniformly on bounded sets of x. Our existence results present as follows:

Lemma 4.1. $c_\varepsilon \to \tilde{\gamma}_m$ *as* $\varepsilon \to 0$.

Lemma 4.2. c_ε *is attained for all small* $\varepsilon > 0$.

Lemma 4.3. $\mathscr{L}_\varepsilon$ *is compact in* $H^1(\mathbb{R}^3, \mathbb{C}^4)$, *for all small* $\varepsilon > 0$.

Proof of Lemma 4.1. Firstly we show that

$$(4.1) \qquad \liminf_{\varepsilon \to 0} c_\varepsilon \geq \tilde{\gamma}_m.$$

Arguing indirectly, assume that $\liminf_{\varepsilon \to 0} c_\varepsilon < \tilde{\gamma}_m$. By the definition of c_ε and Proposition 3.8 we can choose an $e_j \in \mathscr{N}_\varepsilon$ and $\delta > 0$ such that

$$\max_{u \in E_{e_j}} \widetilde{\Phi}_{\varepsilon_j}(u) \leq \tilde{\gamma}_m - \delta$$

as $\varepsilon_j \to 0$. Since $W_\varepsilon(x) \leq m$, the representations of $\widetilde{\Phi}_\varepsilon$ and $\widetilde{\mathscr{T}}_m$ imply that $\widetilde{\Phi}_\varepsilon(u) \geq \widetilde{\mathscr{T}}_m(u)$ for all $u \in E$ and ε small. Note also that $\tilde{\gamma}_m \leq J_m(e_j) \leq \max_{u \in E_{e_j}} \widetilde{\mathscr{T}}_m(u)$. Therefore we get, for all ε_j small,

$$\tilde{\gamma}_m - \delta \geq \max_{u \in E_{e_j}} \widetilde{\Phi}_{\varepsilon_j}(u) \geq \max_{u \in E_{e_j}} \widetilde{\mathscr{T}}_m(u) \geq \tilde{\gamma}_m,$$

a contradiction.

We now turn to prove the desired conclusion. Set $W^0(x) = m - W(x)$ and $W_\varepsilon^0(x) = W^0(\varepsilon x)$. Then

$$(4.2) \qquad \widetilde{\Phi}_\varepsilon(u) = \widetilde{\mathscr{T}}_m(u) + \int W_\varepsilon^0(x) G(|u|).$$

In virtue of Lemma 3.3, let $u = u^+ + u^- \in \mathscr{K}_m$ such that $\widetilde{\mathscr{T}}_m(u) = \tilde{\gamma}_m$ and set $e = u^+$. Surely, $e \in \mathscr{M}_m$, $\mathscr{J}_m(e) = u^-$ and $J_m(e) = \tilde{\gamma}_m$. There is a unique $t_\varepsilon > 0$ such that $t_\varepsilon e \in \mathscr{N}_\varepsilon$ and one has

$$(4.3) \qquad c_\varepsilon \leq I_\varepsilon(t_\varepsilon e).$$

By Lemma 3.9 t_ε is bounded. Hence, without loss of generality we can assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. Using (3.5), we infer

$$\frac{1}{2}\|v_\varepsilon\|^2 + (I) = \widetilde{\Phi}_\varepsilon(w_\varepsilon) - \widetilde{\Phi}_\varepsilon(z_\varepsilon)$$

$$= \widetilde{\mathscr{T}}_m(w_\varepsilon) - \widetilde{\mathscr{T}}_m(z_\varepsilon) + \int W_\varepsilon^0(x)\big(G(|w_\varepsilon|) - G(|z_\varepsilon|)\big)$$

where, setting

$$z_\varepsilon = t_\varepsilon e + \mathscr{J}_m(t_\varepsilon e), \;\; w_\varepsilon = t_\varepsilon e + h_\varepsilon(t_\varepsilon e), \;\; v_\varepsilon = z_\varepsilon - w_\varepsilon,$$

$$(I) := \int_0^1 (1-s)\big(\mathscr{F}_\lambda''(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon] + \Psi_\varepsilon''(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\big) ds.$$

Taking into account that

$$\int W_\varepsilon^0(x)\big(G(|w_\varepsilon|) - G(|z_\varepsilon|)\big)$$

$$= -\Re \int W_\varepsilon^0(x) g(|z_\varepsilon|) z_\varepsilon \cdot \overline{v_\varepsilon} + \int_0^1 (1-s)\mathscr{G}_m''(z_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon] \, ds$$

$$- \int_0^1 (1-s)\Psi_\varepsilon''(z_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon] \, ds,$$

setting

$$(II) := \int_0^1 (1-s)\Psi_\varepsilon''(z_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon] \, ds,$$

following Remark 3.10, one has

$$\frac{1}{2}\|v_\varepsilon\|^2 + (I) + (II) \leq -\Re \int W_\varepsilon^0(x)g(|z_\varepsilon|)z_\varepsilon \cdot \overline{v_\varepsilon} .$$

By noticing that $0 \leq P_\varepsilon^0(x) \leq m$, $(II) \geq 0$ and

$$\left|\mathscr{F}_\lambda''(w_\varepsilon + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\right| \leq \frac{1}{2}\|v_\varepsilon\|^2 ,$$

we deduce that

(4.4) $$\frac{1}{4}\|v_\varepsilon\|^2 \leq \int W_\varepsilon^0(x)g(|z_\varepsilon|)|z_\varepsilon| \cdot |v_\varepsilon| .$$

Since $t_\varepsilon \to t_0$, it is clear that $\{z_\varepsilon\}$, $\{w_\varepsilon\}$ and $\{v_\varepsilon\}$ are bounded and, particularly, for $q \in [2,3]$

$$\limsup_{r\to\infty} \int_{|x|>r} |z_\varepsilon|^q = 0.$$

Now we infer

$$\int \left(W_\varepsilon^0(x)\right)^{q/(q-1)} |u_\varepsilon|^q$$
$$= \left(\int_{|x|\leq r} + \int_{|x|>r}\right) W_\varepsilon^0(x)^{q/(q-1)}|u_\varepsilon|^q$$
$$\leq \int_{|x|\leq r} \left(W_\varepsilon^0(x)\right)^{q/(q-1)} |u_\varepsilon|^q + m^{q/(q-1)} \int_{|x|>r} |u_\varepsilon|^q$$
$$= o(1)$$

as $\varepsilon \to 0$. Thus by (4.4) one has $\|v_\varepsilon\|^2 \to 0$, that is, $h_\varepsilon(t_\varepsilon e) \to \mathscr{J}_m(t_0 e)$. Consequently,

$$\int W_\varepsilon^0(x)G(|w_\varepsilon|) \to 0$$

as $\varepsilon \to 0$. This, jointly with (4.2), shows

$$\widetilde{\Phi}_\varepsilon(w_\varepsilon) = \widetilde{\mathscr{T}_m}(w_\varepsilon) + o(1) = \widetilde{\mathscr{T}_m}(z_\varepsilon) + o(1),$$

that is,

$$I_\varepsilon(t_\varepsilon e) = J_m(t_0 e) + o(1)$$

as $\varepsilon \to 0$. Then, since

$$J_m(t_0 e) \leq \max_{v \in E_e} \widetilde{\mathscr{T}_m}(v) = J_m(e) = \tilde{\gamma}_m,$$

we obtain by using (4.1) and (4.3)

$$\tilde{\gamma}_m \leq \lim_{\varepsilon\to 0} c_\varepsilon \leq \lim_{\varepsilon\to 0} I_\varepsilon(t_\varepsilon e) = J_m(t_0 e) \leq \tilde{\gamma}_m,$$

hence, $c_\varepsilon \to \tilde{\gamma}_m$. $\qquad\qquad \square$

Proof of Lemma 4.2. Given $\varepsilon > 0$, let $\{u_n\} \subset \mathscr{N}_\varepsilon$ be a minimization sequence: $I_\varepsilon(u_n) \to c_\varepsilon$. By the Ekeland variational principle we can assume that $\{u_n\}$ is in fact a $(PS)_{c_\varepsilon}$-sequence for I_ε on E^+ (see [26, 31]). Then $w_n = u_n + h_\varepsilon(u_n)$ is a $(PS)_{c_\varepsilon}$-sequence for $\widetilde{\Phi}_\varepsilon$ on E. It is clear that $\{w_n\}$ is bounded, hence is a $(C)_{c_\varepsilon}$-sequence. We can assume without loss of generality that $w_n \rightharpoonup w_\varepsilon = w_\varepsilon^+ + w_\varepsilon^- \in \mathscr{K}_\varepsilon$ in E. If $w_\varepsilon \neq 0$ then $\widetilde{\Phi}_\varepsilon(w_\varepsilon) = c_\varepsilon$. So we are going to show that $w_\varepsilon \neq 0$ for all small $\varepsilon > 0$.

To this end, take $\limsup_{|x| \to \infty} W(x) < \kappa < m$ and define

$$W^\kappa(x) = \min\{\kappa, W(x)\}.$$

Set $A := \{x \in \mathbb{R}^3 : W(x) > \kappa\}$ and $A_\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon x \in A\}$. Following (P_0), A_ε is a bounded set for any fixed ε. Consider the functional

$$\widetilde{\Phi}_\varepsilon^\kappa(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \mathscr{F}_\lambda(u) - \int W_\varepsilon^\kappa(x) G(|u|)$$

and as before define correspondingly $h_\varepsilon^\kappa : E^+ \to E^-$, $I_\varepsilon^\kappa : E^+ \to \mathbb{R}$, $\mathscr{N}_\varepsilon^\kappa$, c_ε^κ and so on. As done in the proof of Lemma 4.1,

(4.5) $$\lim_{\varepsilon \to 0} c_\varepsilon^\kappa = \tilde{\gamma}_\kappa.$$

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L_{loc}^q for $q \in [1,3)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^3$. Let $t_n > 0$ be such that $t_n u_n \in \mathscr{N}_{\varepsilon_j}^\kappa$. Since $u_n \in \mathscr{N}_\varepsilon$, it is not difficult to see $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. Remark that $h_{\varepsilon_j}^\kappa(t_n u_n) \rightharpoonup 0$ in E and $h_{\varepsilon_j}^\kappa(t_n u_n) \to 0$ in L_{loc}^q for $q \in [1,3)$ as $n \to \infty$ (see [1]). Moreover, we remind that

$$\widetilde{\Phi}_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^\kappa(t_n u_n)) \leq I_{\varepsilon_j}(t_n u_n) \leq I_{\varepsilon_j}(u_n).$$

So, we obtain

$$c_{\varepsilon_j}^\kappa \leq I_{\varepsilon_j}^\kappa(t_n u_n) = \widetilde{\Phi}_{\varepsilon_j}^\kappa(t_n u_n + h_{\varepsilon_j}^\kappa(t_n u_n))$$

$$= \widetilde{\Phi}_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^\kappa(t_n u_n)) + \int \big(P_{\varepsilon_j}(x) - P_{\varepsilon_j}^\kappa(x)\big) G\big(|t_n u_n + h_{\varepsilon_j}^\kappa(t_n u_n)|\big)$$

$$\leq I_{\varepsilon_j}(u_n) + \int_{A_{\varepsilon_j}} \big(P_{\varepsilon_j}(x) - P_{\varepsilon_j}^\kappa(x)\big) G\big(|t_n u_n + h_{\varepsilon_j}^\kappa(t_n u_n)|\big)$$

$$= c_{\varepsilon_j} + o(1)$$

as $n \to \infty$. Hence $c_{\varepsilon_j}^\kappa \leq c_{\varepsilon_j}$. By (4.5), letting $j \to \infty$ yields

$$\tilde{\gamma}_\kappa \leq \tilde{\gamma}_m,$$

which contradicts $\tilde{\gamma}_m < \tilde{\gamma}_\kappa$. $\qquad\square$

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Proof of Lemma 4.3. Since $\mathscr{L}_\varepsilon \subset B_\Lambda$ for all small $\varepsilon > 0$, assume by contradiction that, for some $\varepsilon_j \to 0$, $\mathscr{L}_{\varepsilon_j}$ is not compact in E. Let $u_n^j \in \mathscr{L}_{\varepsilon_j}$ with $u_n^j \rightharpoonup 0$ as $n \to \infty$. As done in proving the Lemma 4.2, one gets a contradiction.

Now let $\{u_n\} \subset \mathscr{L}_\varepsilon$ such that $u_n \to u$ in E. We recall that $H_\omega = i\alpha \cdot \nabla - a\beta + \omega$, by

$$H_\omega u_n = \lambda V_{u_n} \beta u_n + W_\varepsilon(x) g(|u_n|) u_n$$

and

$$H_\omega u = \lambda V_u \beta u + W_\varepsilon(x) g(|u|) u$$

we deduce

$$(4.6) \quad |H_\omega(u_n - u)|_2 \leq \lambda \big| V_{u_n} u_n - V_u u \big|_2 + \big| W_\varepsilon \cdot \big(g(|u_n|) u_n - g(|u|) u \big) \big|_2 .$$

Invoking Lemma 2.7 and $u_n \to u$ in $L^q(\mathbb{R}^3, \mathbb{C}^4)$ for all $q \in [2,3]$, one gets $|H_\omega(u_n - u)|_2 \to 0$ as $n \to \infty$, and that is, $u_n \to u$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$. $\square$

Part 2. Concentration

It is contained in the following lemma. To prove the lemma, it suffices to show that for any sequence $\varepsilon_j \to 0$ the corresponding sequence of solutions $u_j \in \mathscr{L}_{\varepsilon_j}$ converges, up to a shift of x-variable, to a least energy solution of the limit problem (1.4).

Lemma 4.4. *Suppose that ∇W is bounded. There is a maximum point x_ε of $|u_\varepsilon|$ such that $\mathrm{dist}(y_\varepsilon, \mathscr{C}) \to 0$ where $y_\varepsilon = \varepsilon x_\varepsilon$, and for any such x_ε, $v_\varepsilon(x) := u_\varepsilon(x + x_\varepsilon)$ converges to a ground state solution of (1.4) in H^1 as $\varepsilon \to 0$.*

Proof. Let $\varepsilon_j \to 0$, $u_j \in \mathscr{L}_j$, where $\mathscr{L}_j = \mathscr{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument (see [24]) shows that there exist a sequence $\{x_j\} \subset \mathbb{R}^3$ and constant $R > 0$, $\delta > 0$ such that

$$\liminf_{j \to \infty} \int_{B(x_j, R)} |u_j|^2 \geq \delta.$$

Set

$$v_j = u_j(x + x_j),$$

and denoted by $\hat{W}_j(x) = W(\varepsilon_j(x + x_j))$, one easily checks that v_j solves

$$(4.7) \qquad H_\omega v_j - \lambda V_{v_j} \beta v_j = \hat{W}_j \cdot g(|v_j|) v_j,$$

with energy

$$S(v_j) := \frac{1}{2}\big(\|v_j^+\|^2 - \|v_j^-\|^2\big) - \Gamma_\lambda(v_j) - \int \hat{W}_j(x)G(|v_j|)$$

$$= \widetilde{\Phi}_j(v_j) = \Phi_j(v_j) = \Gamma_\lambda(v_j) + \int \hat{W}_j(x)\widehat{G}(|v_j|)$$

$$= c_{\varepsilon_j}.$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L_{loc}^q for $q \in [1,3)$.

We now turn to prove that $\{\varepsilon_j x_j\}$ is bounded. Arguing indirectly we assume $\varepsilon_j |x_j| \to \infty$ and get a contradiction.

Without loss of generality assume $W(\varepsilon_j x_j) \to W_\infty$. By the boundness of ∇W, one sees that $\hat{W}_j(x) \to W_\infty$ uniformly on bounded sets of x. Surely, $m > W_\infty$ by (P_0). Since for any $\psi \in C_c^\infty$

$$0 = \lim_{j\to\infty} \int \big(H_\omega v_j - \lambda V_{v_j}\beta v_j - \hat{W}_j g(|v_j|)v_j\big)\bar\psi$$

$$= \lim_{j\to\infty} \int \big(H_\omega v - \lambda V_v\beta v - W_\infty g(|v|)v\big)\bar\psi,$$

hence v solves

$$i\alpha \cdot \nabla v - a\beta v + \omega v - \lambda V_v\beta v = W_\infty g(|v|)v.$$

Therefore,

$$S_\infty(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) - \Gamma_\lambda(v) - W_\infty \int G(|v|) \geq \tilde\gamma_{W_\infty}.$$

It follows from $m > P_\infty$, by Proposition 3.8, one has $\tilde\gamma_m < \tilde\gamma_{W_\infty}$. Moreover, by the Fatou's lemma,

$$\lim_{j\to\infty} \int \hat{W}_j(x)\widehat{G}(|v_j|) \geq \int W_\infty \widehat{G}(|v|).$$

Consequently, noting that $\liminf_{j\to\infty} \Gamma_\lambda(v_j) \geq \Gamma_\lambda(v)$, we have

$$\tilde\gamma_m < \tilde\gamma_{W_\infty} \leq S_\infty(v) \leq \lim_{j\to\infty} c_{\varepsilon_j} = \tilde\gamma_m,$$

a contradiction.

Thus $\{\varepsilon_j x_j\}$ is bounded. And hence, we can assume $y_j = \varepsilon_j x_j \to y_0$. Then v solves

(4.8) $$i\alpha \cdot \nabla v - a\beta v + \omega v - \lambda V_v\beta v = W(y_0)g(|v|)v.$$

Since $W(y_0) \leq m$, we obtain

$$S_0(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) - \Gamma_\lambda(v) - W(y_0)\int G(|v|) \geq \tilde\gamma_{W(y_0)} \geq \tilde\gamma_m.$$

Again, by Fatou's lemma, we have

$$S_0(v) = \int P(y_0)\widehat{G}(|v|) + \Gamma_\lambda(v) \le \lim_{j\to\infty} c_{\varepsilon_j} = \tilde{\gamma}_m.$$

Therefore, $\gamma_{P(y_0)} = \gamma_m$, which implies $y_0 \in \mathscr{C}$ by Proposition 3.8. By virtue of Lemma 3.11 and (3.9) it is clear that one may assume that $x_j \in \mathbb{R}^3$ is a maximum point of $|u_j|$. Moreover, from the above argument we readily see that, any sequence of such points satisfies $y_j = \varepsilon_j x_j$ converging to some point in $\mathscr{C}$ as $j \to \infty$.

In order to prove $v_j \to v$ in E, recall that as the argument shows

$$\lim_{j\to\infty} \int \hat{W}_j(x)\widehat{G}(|v_j|) = \int W(y_0)\widehat{G}(|v|).$$

By (G_2) and the decay of v, using the Brezis-Lieb lemma, one obtains $|v_j - v|_\sigma \to 0$, then $|v_j^\pm - v^\pm|_\sigma \to 0$ by (2.4). Denote $z_j = v_j - v$. Remark that $\{z_j\}$ is bounded in E and $z_j \to 0$ in L^σ, therefore $z_j \to 0$ in L^q for all $q \in (2, 3)$. The scalar product of (4.7) with z_j^+ yields

$$\langle v_j^+, z_j^+ \rangle = o(1).$$

Similarly, using the decay of v together with the fact that $z_j^\pm \to 0$ in L^q_{loc} for $q \in [1, 3)$, it follows from (4.8) that

$$\langle v^+, z_j^+ \rangle = o(1).$$

Thus

$$\|z_j^+\| = o(1),$$

and the same arguments show

$$\|z_j^-\| = o(1),$$

we then get $v_j \to v$ in E, and the arguments in Lemma 4.3 show that $v_j \to v$ in H^1. $\qquad\square$

Part 3. Exponential decay

See the following Proposition 4.6. For the later use denote $D = i\alpha \cdot \nabla$ and, for $u \in \mathscr{L}_\varepsilon$, rewrite (2.9) as

$$Du = a\beta u - \omega u + \lambda V_u \beta u + W_\varepsilon(x)g(|u|)u.$$

Acting the operator D on the two sides and noting that $D^2 = -\Delta$, we get

(4.9)
$$\begin{aligned} \Delta u = &\left(a + \lambda V_u\right)^2 u - \left(\omega - W_\varepsilon \cdot g(|u|)\right)^2 u \\ &- D\left(\lambda V_u + W_\varepsilon \cdot g(|u|)\right)u. \end{aligned}$$

Now define

$$\operatorname{sgn} u = \begin{cases} \dfrac{\bar{u}}{|u|} & \text{if } u \neq 0 \,, \\ 0 & \text{if } u = 0 \,. \end{cases}$$

By Kato's inequality [10], there holds

$$\Delta |u| \geq \Re\big[\Delta u(\operatorname{sgn} u)\big] \,.$$

Note that

$$\Re\big[D\big(\lambda V_u + W_\varepsilon \cdot g(|u|)\big) u(\operatorname{sgn} u)\big] = 0 \,.$$

Then, we obtain

$$(4.10) \qquad \Delta |u| \geq \big(a + \lambda V_u\big)^2 |u| - \big(\omega - W_\varepsilon \cdot g(|u|)\big)^2 |u| \,.$$

To get the uniformly decay estimate for the semi-classical states, we first need the following result:

Lemma 4.5. *Let v_ε and V_{v_ε} be given in the proof of Lemma 4.4. Then $|v_\varepsilon(x)|$ and $|V_{v_\varepsilon}(x)|$ vanish at infinity uniformly in $\varepsilon > 0$ small.*

Due to (4.10), we remark that Lemma 4.5 makes it feasible to choose $R > 0$ (independent of ε) such that

$$\Delta |v_\varepsilon| \geq \frac{a^2 - \omega^2}{2} |v_\varepsilon| \quad \text{for } |x| \geq R \,.$$

And at this point, applying the maximum principle (see [27]), we easily have

Proposition 4.6. *Let $v_\varepsilon \in E$ be given in the proof of Lemma 4.4, then v_ε exponentially decays at infinity uniformly in $\varepsilon > 0$ small. More specifically, there exist $C, c > 0$ independent of ε such that*

$$|v_\varepsilon(x)| \leq C e^{-c|x|} \,.$$

Consequently, we infer that

$$|u_\varepsilon(x)| \leq C e^{-c|x - x_\varepsilon|} \,.$$

Now, we turn to prove Lemma 4.5. To begin with, we remind that (4.10) together with the regularity results for u (see Lemma 3.11) implies there is $M > 0$ (independent of ε) satisfying

$$\Delta |u| \geq -M |u| \,.$$

It then follows from the sub-solution estimate [22, 29] that

$$(4.11) \qquad |u(x)| \leq C_0 \int_{B_1(x)} |u(y)| dy$$

with $C_0 > 0$ independent of x, ε and $u \in \mathscr{L}_\varepsilon$.

Proof of Lemma 4.5. Assume by contradiction that there exist $\delta > 0$ and $x_\varepsilon \in \mathbb{R}^3$ with $|x_\varepsilon| \to \infty$ such that

$$\delta \le |v_\varepsilon(x_\varepsilon)| \le C_0 \int_{B_1(x_\varepsilon)} |v_\varepsilon(y)| dy \,.$$

Since $v_\varepsilon \to v$ in E, we obtain, as $\varepsilon \to 0$,

$$\delta \le C_0 \Big(\int_{B_1(x_\varepsilon)} |v_\varepsilon|^2 \Big)^{1/2}$$
$$\le C_0 \Big(\int |v_\varepsilon - v|^2 \Big)^{1/2} + C_0 \Big(\int_{B_1(x_\varepsilon)} |v|^2 \Big)^{1/2} \to 0,$$

a contradiction. Now, jointly with (3.7), one sees also $|V_{v_\varepsilon}(x)| \to 0$ as $|x| \to \infty$ uniformly in $\varepsilon > 0$ small. $\square$

With the above arguments, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Going back to system (1.3), with the variable substitution: $x \mapsto x/\varepsilon$, Lemma 4.2 jointly with Lemma 3.11 and the elliptic regularity shows that, for all $\varepsilon > 0$ small, Eq.(1.3) has at least one ground state solution $(\varphi_\varepsilon, \phi_\varepsilon) \in \cap_{q \ge 2} W^{1,q} \times C^2$. Moreover, by Lemma 4.3 and Lemma 2.7, one easily checks the compactness of the ground states.

Assume additionally ∇W is bounded, Lemma 4.4 is nothing but the concentration result. And finally, Proposition 4.6 gives the decay estimate. $\square$

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